

Stress Tensor Fluctuations and Passive Quantum Gravity

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The quantum fluctuations of the stress tensor of a quantum field are discussed, as are the resulting space–time metric fluctuations. Passive quantum gravity is an approximation in which gravity is not directly quantized, but fluctuations of the space–time geometry are driven by stress tensor fluctuations. We discuss a decomposition of the stress tensor correlation function into three parts, and consider the physical implications of each part. The operational significance of metric fluctuations and the possible limits of validity of semiclassical gravity are discussed.

KEY WORDS: quantum gravity; metric fluctuations; stress tensor fluctuations.

1. INTRODUCTION

The essential divide between classical gravity and the various quantum versions of gravity theory is crossed when the space–time geometry ceases to be fixed, but rather undergoes fluctuations. In a complete theory of quantum gravity, one expects these fluctuations to arise both from the quantum nature of gravity itself and from the quantum fluctuations of matter fields which act as the source of gravity. The former are “active” (or spontaneous) fluctuations, whereas the latter are “passive” (or induced) fluctuations. A complete description of active space–time metric fluctuations would require a full quantum theory of gravity. However, it is possible to use linearized quantum gravity to describe a variety of nontrivial phenomena, including quantum fluctuations of the lightcone (Ford, 1995). In this paper, we will focus upon the passive metric fluctuations. Thus the gravitational field will not be quantized, but nonetheless will undergo quantum fluctuations driven by matter fields. This is what we mean by the phrase “passive quantum gravity.”

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The key to understanding passive quantum gravity is an analysis of the fluctuations of the stress tensor of a quantized field, which will be the principle topic of this paper. Most of our discussion will deal with quantum fields on an approximately flat background. Fluctuations of the quantum stress tensor were discussed in literature (del Campo and Ford, 1988; Ford, 1982; Kuo and Ford, 1993; Wu and Ford, 1999), using an approach based on normal ordering, which will be discussed in more detail below. Other authors (Calzetta *et al.*, 1997; Calzetta and Hu, 1993, 1995; Martin and Verdaguier, 1999) have discussed stress tensor fluctuations in the context of cosmology.

In this paper, we will discuss a useful decomposition of the product of stress tensor operators into three terms, a fully normal ordered term, a cross term, and a vacuum term. The possible physical implications of each of these terms will be considered in succession. In particular, we will discuss how the cross term is responsible for the quantum fluctuations of radiation pressure when a laser beam impinges upon a mirror, a potentially observable effect. We will also present some new results concerning the pure vacuum term. We calculate both the stress tensor correlation function and the resulting metric tensor correlation function in the Minkowski vacuum state, and show that the latter quantity can be expressed as total derivatives of a scalar function. We then discuss the operational meaning of metric fluctuations as the Brownian motion of test particles. We conclude with some remarks on the likely range of validity of semiclassical gravity in which metric fluctuations are ignored.

2. THE STRESS TENSOR CORRELATION FUNCTION

The basic object of interest is the quantum stress tensor operator, $T_{\mu\nu}(x)$. However, this object is defined only after a renormalization. That is, the formal expectation value of $T_{\mu\nu}(x)$ in any quantum state is divergent. Fortunately, the divergence is a c-number, so the renormalization is state-independent. The details of this procedure on a curved background can be rather elaborate, and are discussed in many references (Birrell and Davies, 1982). For our purposes, it is sufficient to discuss the quantum stress tensor operator in Minkowski space–time. In this case, the c-number to be subtracted is simply the expectation value of $T_{\mu\nu}(x)$ in the Minkowski vacuum state, and the renormalized operator is the normal ordered operator:

$$: T_{\mu\nu}(x) := T_{\mu\nu}(x) - \langle T_{\mu\nu}(x) \rangle_0. \quad (1)$$

Here $\langle \rangle_0$ denotes the expectation value in the Minkowski vacuum state.

To discuss stress tensor fluctuations, we must be able to define the correlation function of a pair of renormalized stress tensor operators. If we restrict ourselves to flat space–time and normal ordered stress tensor operators, then we can define

the correlation function as

$$C_{\mu\nu\rho\sigma}(x, x') = \langle : T_{\mu\nu}(x) :: T_{\rho\sigma}(x') : \rangle, \quad (2)$$

where the expectation value is understood to be taken in an arbitrary quantum state. This correlation function can be decomposed into three parts using Wick's theorem. The following identity can be established using this theorem:

$$\begin{aligned} : \phi_1 \phi_2 :: \phi_3 \phi_4 : &= : \phi_1 \phi_2 \phi_3 \phi_4 : + : \phi_1 \phi_3 : \langle \phi_2 \phi_4 \rangle_0 + : \phi_1 \phi_4 : \langle \phi_2 \phi_3 \rangle_0 \\ &+ : \phi_2 \phi_3 : \langle \phi_1 \phi_4 \rangle_0 + : \phi_2 \phi_4 : \langle \phi_1 \phi_3 \rangle_0 + \langle \phi_1 \phi_3 \rangle_0 \langle \phi_2 \phi_4 \rangle_0 \\ &+ \langle \phi_1 \phi_4 \rangle_0 \langle \phi_2 \phi_3 \rangle_0, \end{aligned} \quad (3)$$

where the ϕ_i are free bosonic fields. In the remainder of this paper, we will assume that our stress tensor operators are those of free bosonic fields, and hence can be expressed as quadratic forms in the ϕ_i .

We can now express the correlation function as

$$C^{\mu\nu\rho\sigma}(x, x') = C_{(N)}^{\mu\nu\rho\sigma}(x, x') + C_{(cross)}^{\mu\nu\rho\sigma}(x, x') + C_{(V)}^{\mu\nu\rho\sigma}(x, x'). \quad (4)$$

Here

$$C_{(N)}^{\mu\nu\rho\sigma}(x, x') = \langle : T_{\mu\nu}(x) T_{\rho\sigma}(x') : \rangle \quad (5)$$

is a fully normal ordered operator, and

$$C_{(V)}^{\mu\nu\rho\sigma}(x, x') = \langle : T_{\mu\nu}(x) :: T_{\rho\sigma}(x') : \rangle_0 \quad (6)$$

is a pure vacuum term. $C_{(cross)}^{\mu\nu\rho\sigma}(x, x')$ is a cross term which is expressible as a sum of products of normal ordered quadratic operators and vacuum expectation values of quadratic operators, that is, products of the form of the middle four terms in Eq. (3). The fully normal ordered term is state-dependent and finite in the coincidence limit, $x' \rightarrow x$. The pure vacuum term is singular in this limit, but is state-independent. However, the cross term is both state-dependent and singular as $x' \rightarrow x$. Thus it is not possible to render $C^{\mu\nu\rho\sigma}(x, x')$ finite by a state-independent subtraction, as it is $\langle T_{\mu\nu}(x) \rangle$.

As we vary the quantum state to increase the mean energy density $\langle : \rho : \rangle = \langle : T_{tt} : \rangle$, the fully normal ordered term will scale as $\langle : \rho : \rangle^2$, the cross term as $\langle : \rho : \rangle$, and the vacuum term does not change. Thus, in the limit of highly excited quantum states, the normal ordered term will dominate. However, its contribution to the stress tensor fluctuations,

$$C^{\mu\nu\rho\sigma}(x, x') - \langle T_{\mu\nu}(x) \rangle \langle T_{\rho\sigma}(x') \rangle \quad (7)$$

need not grow any faster than that of the cross term. The physical implications of each of the three terms will be discussed in turn in the following sections.

3. THE FULLY NORMAL ORDERED TERM

This term, as noted above, has the feature that it is finite in the coincidence limit. If this were the only term present in the correlation function, then one could meaningfully discuss the fluctuations in local stress tensor components, such as the energy density. This approach was used by Kuo and Ford (1993), where only the fully normal ordered term was retained, and a dimensionless measure of the local energy density fluctuations was defined:

$$\Delta = \frac{\langle : \rho^2 : \rangle - \langle : \rho : \rangle^2}{\langle : \rho^2 : \rangle}. \quad (8)$$

In the case of a coherent state, $\Delta = 0$, so there are no fluctuations in the local energy density by this measure. However, in nonclassical states, such as a squeezed vacuum state or a Casimir vacuum state, one can have Δ of order unity. Similar results were found by Phillips and Hu (1997) for the vacuum energy density in symmetric, curved space-times.

These results can be summarized by saying that $C_{(N)}^{\mu\nu\rho\sigma}(x, x')$ describes a fluctuating local energy density. In the classical limit, these fluctuations vanish, but for nonclassical states, the fluctuations in the local energy density can be at least as large as the mean energy density.

4. THE CROSS TERM

The simple picture of stress tensor fluctuations based upon $C_{(N)}^{\mu\nu\rho\sigma}(x, x')$ alone is not complete, in part because of the existence of the cross term, $C_{(cross)}^{\mu\nu\rho\sigma}(x, x')$. This term depends upon the quantum state, but is singular when $x' \rightarrow x$. Thus it is not possible to define a local quantity analogous to Δ which describes the effects of this term. However, this does not mean that the cross term is devoid of physical meaning. On the contrary, it is essential for understanding such phenomena as the quantum fluctuations of radiation pressure.

4.1. Finiteness of Integrals of the Cross Term

The singularity of the cross term need not be a concern if observable quantities, which are space and time integrals, can be defined. The cross term goes as $(x - x')^{-4}$ as $x' \rightarrow x$. At first sight, this is not an integrable singularity. However, it is in fact possible to define the relevant integrals by an integration by parts procedure. The basic idea can be illustrated as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} dt_1 dt_2 f(t_1) f(t_2) \frac{1}{(t_1 - t_2)^4} &= -\frac{1}{12} \int_{-\infty}^{\infty} dt_1 dt_2 f(t_1) f(t_2) \frac{\partial^4}{\partial t_1^2 \partial t_2^2} \ln [(t_1 - t_2)^2 \mu^2] \\ &= -\frac{1}{12} \int_{-\infty}^{\infty} dt_1 dt_2 \ddot{f}(t_1) \ddot{f}(t_2) \ln [(t_1 - t_2)^2 \mu^2], \quad (9) \end{aligned}$$

where μ is an arbitrary constant. We have assumed that the function $f(t)$ vanishes as $|t| \rightarrow \infty$, so the surface terms in the integration by parts vanish. The effect of this manipulation is to replace the apparently nonintegrable singularity in the first integral by a mild, integrable singularity in the final integral. This trick has been employed by various authors under the labels “generalized principal value integration” (Davies *et al.*, 1990; Davies and Davies, 1989) or “differential regularization” (Freedman *et al.*, 1992). Because the quantum state describes a distribution of energy which is limited in time, the normal ordered factors in $C_{(cross)}^{\mu\nu\rho\sigma}(x, x')$ vanish in both the past and the future, allowing the surface terms in the integrations to be dropped.

4.2. Quantum Fluctuations of Radiation Pressure

Classically, a beam of light falling on a mirror exerts a force and the force can be written as the integral of the Maxwell stress tensor. When we treat this problem quantum mechanically, then the force undergoes fluctuations. This is a necessary consequence of the fact that physically realizable quantum states are not eigenstates of the stress tensor operator. These radiation pressure fluctuations play an important role in limiting the sensitivity of laser interferometer detectors of gravitational radiation, as was first analyzed by Caves (1980, 1981). His approach was based on the statistical fluctuations of photon numbers in a coherent state. Recently, we (Wu and Ford, 2001) have shown how this phenomenon can be understood in the context of the quantum stress tensor. Here we will give a brief summary of this treatment.

Consider a mirror of mass m which is oriented perpendicularly to the x -direction. If the mirror is at rest at time $t = 0$, then at time $t = \tau$ its velocity in the x -direction is given classically by

$$v = \frac{1}{m} \int_0^\tau dt \int_A da T_{xx}, \quad (10)$$

where T_{ij} is the Maxwell stress tensor, and $\int_A da$ denotes an integration over the surface of the mirror. Here we assume that there is radiation present on one side of the mirror only. Otherwise, Eq. (10) would involve a difference in T_{xx} across the mirror. When the radiation field is quantized, T_{ij} is replaced by the normal ordered operator $:T_{ij}:$, and Eq. (10) becomes a Langevin equation. The dispersion in the mirror’s velocity becomes

$$\begin{aligned} \langle \Delta v^2 \rangle &= \frac{1}{m^2} \int_0^\tau dt \int_0^\tau dt' \int_A da \int_A da' [:T_{xx}(x) :: T_{xx}(x') : \\ &\quad - \langle :T_{xx}(x) : \rangle \langle :T_{xx}(x') : \rangle]. \end{aligned} \quad (11)$$

We now assume that the photons are in a single mode coherent state, so that the fully normal ordered term gives no contribution. We are also only interested in

the changes in $\langle \Delta v^2 \rangle$ due to the radiation. Thus we can subtract off the Minkowski vacuum contribution $\langle \Delta v^2 \rangle_0$ and ignore the pure vacuum term. Now the entire contribution to the mirror's velocity fluctuations comes from the cross term:

$$\langle \Delta v^2 \rangle = \frac{1}{m^2} \int_0^\tau dt \int_0^\tau dt' \int_A da \int_A da' \langle T_{xx}(x) T_{xx}(x') \rangle_{cross}. \quad (12)$$

The relevant component of the stress tensor is (Lorentz–Heaviside units are used here.)

$$T_{xx} = \frac{1}{2} (E_y^2 + E_z^2 + B_y^2 + B_z^2). \quad (13)$$

We now assume that a linearly polarized plane wave is normally incident and is perfectly reflected by the mirror. Take the polarization vector to be in the y -direction, so that $E_z = B_y = 0$. At the location of the mirror, $E_y = 0$, and only B_z contributes to the stress tensor. Thus, when we apply Eq. (3) to find $\langle T_{xx}(x) T_{xx}(x') \rangle_{cross}$, the only nonzero quadratic normal-ordered product will be $\langle : B_z(x) B_z(x') : \rangle$. The result is

$$\langle T_{xx}(x) T_{xx}(x') \rangle_{cross} = \langle : B_z(x) B_z(x') : \rangle \langle B_z(x) B_z(x') \rangle_0. \quad (14)$$

The vacuum magnetic field two-point function in the presence of a perfectly reflecting plane at $z = 0$ is given by

$$\langle B_z(t_1, \mathbf{x}_1) B_z(t_2, \mathbf{x}_2) \rangle_0 = \langle B_z(t_1, \mathbf{x}_1) B_z(t_2, \mathbf{x}_2) \rangle_{E0} + \langle B_z(t_1, \mathbf{x}_1) B_z(t_2, \mathbf{x}_2) \rangle_{I0}. \quad (15)$$

The first term is the two-point function for empty space,

$$\langle B_z(t_1, \mathbf{x}_1) B_z(t_2, \mathbf{x}_2) \rangle_{E0} = \frac{(t_1 - t_2) + |\mathbf{x}_1 - \mathbf{x}_2|^2 - 2(z_1 - z_2)^2}{\pi^2 [(t_1 - t_2)^2 - |\mathbf{x}_1 - \mathbf{x}_2|^2]^3}. \quad (16)$$

The second term is an image term

$$\langle B_z(t_1, \mathbf{x}_1) B_z(t_2, \mathbf{x}_2) \rangle_{I0} = \langle B_z(t_1, \mathbf{x}_1) B_z(t_2, \mathbf{x}_2) \rangle_{E0} \Big|_{z_2 \rightarrow -z_2}. \quad (17)$$

Both terms give equal contributions to the radiation pressure fluctuations on a mirror located at $z = 0$.

We can see that the integrand in Eq. (12) is singular when the points (t_1, \mathbf{x}_1) and (t_2, \mathbf{x}_2) are lightlike separated from one another. However, this singularity can be handled either by the integration by parts method of the previous subsection, or equivalently by treating the integrals as containing higher order poles. The result is (see Wu and Ford (2001) for details)

$$\langle \Delta v^2 \rangle = 4 \frac{A \omega \rho}{m^2} \tau, \quad (18)$$

where A is the illuminated area of the mirror, and ρ is the mean energy density in the laser beam.

As noted above, this result can be found from considerations of photon number fluctuations. However, in an approach based upon the quantum stress tensor, it arises solely from the cross term. Laser interferometer detectors of gravity waves will eventually have to contend with radiation pressure fluctuations as a noise source. At that point, it is reasonable to expect that these fluctuations will be observed experimentally for the first time. Such an observation would constitute experimental proof of the reality of the cross term.

It is of interest to note that if the quantum state is taken to be a photon number eigenstate, rather than a coherent state, then the fully normal ordered term gives a nonzero contribution. However, this contribution is such as to exactly cancel the contribution coming from the cross term, leaving no radiation pressure fluctuations (Wu and Ford, 2001).

5. THE PURE VACUUM TERM

The piece of the stress tensor correlation function which is the most difficult to interpret is the pure vacuum part, $C_{(V)}^{\mu\nu\rho\sigma}(x, x')$. This term is not only highly divergent in the coincidence limit, but is always present. Any physical effects which it produces would have to be very small so as not to have already been observed. In this section, we will show that it can be written as a total derivative.

5.1. Explicit Form for the Electromagnetic (EM) Field

The stress tensor of EM field is

$$T_{\mu\nu} = F_{\mu}{}^{\rho} F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}, \quad (19)$$

where $F_{\alpha\beta} = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha}$. Expand the stress tensor in terms of the vector potential A_{μ} to find

$$\begin{aligned} T_{\mu\nu} = & \partial_{\mu} A^{\rho} \partial_{\nu} A_{\rho} + \partial^{\rho} A_{\mu} \partial_{\rho} A_{\nu} - \partial^{\rho} A_{\mu} \partial_{\nu} A_{\rho} - \partial_{\mu} A^{\rho} \partial_{\rho} A_{\nu} \\ & - \frac{1}{2} g_{\mu\nu} (\partial_{\alpha} A_{\beta} \partial^{\alpha} A^{\beta} - \partial_{\alpha} A_{\beta} \partial^{\beta} A^{\alpha}). \end{aligned} \quad (20)$$

In the Lorentz gauge,

$$\langle A^{\mu}(x) A^{\nu}(x') \rangle_0 = -g^{\mu\nu} D(x - x'), \quad (21)$$

where

$$D = D(x - x') = \frac{1}{4\pi^2(x - x')^2} \quad (22)$$

is the Hadamard (symmetric two-point) function for the massless scalar field. We can see from Eq. (3) that

$$\begin{aligned} \langle : A_\mu(x) A_\nu(x) :: A_\rho(x') A_\sigma(x') : \rangle_0 &= \langle A_\mu(x) A_\rho(x') \rangle_0 \langle A_\nu(x) A_\sigma(x') \rangle_0 \\ &+ \langle A_\mu(x) A_\sigma(x') \rangle_0 \langle A_\nu(x) A_\rho(x') \rangle_0. \end{aligned} \quad (23)$$

We can now combine these various relations to write, after some calculation, an expression for the vacuum stress tensor correlation function

$$\begin{aligned} C_{(V)}^{\mu\nu\sigma\lambda}(x, x') &= 4(\partial_\mu \partial_\nu D)(\partial_\sigma \partial_\lambda D) + 2g_{\mu\nu}(\partial_\sigma \partial_\alpha D)(\partial_\lambda \partial^\alpha D) \\ &+ 2g_{\sigma\lambda}(\partial_\mu \partial_\alpha D)(\partial_\nu \partial^\alpha D) - 2g_{\mu\sigma}(\partial_\nu \partial_\alpha D)(\partial_\lambda \partial^\alpha D) \\ &- 2g_{\nu\sigma}(\partial_\mu \partial_\alpha D)(\partial_\lambda \partial^\alpha D) - 2g_{\nu\lambda}(\partial_\mu \partial_\alpha D)(\partial_\sigma \partial^\alpha D) \\ &- 2g_{\mu\lambda}(\partial_\nu \partial_\alpha D)(\partial_\sigma \partial^\alpha D) + (g_{\mu\sigma} g_{\nu\lambda} + g_{\nu\sigma} g_{\mu\lambda} \\ &- g_{\mu\nu} g_{\sigma\lambda})(\partial_\rho \partial_\alpha D)(\partial^\rho \partial^\alpha D). \end{aligned} \quad (24)$$

A similar result for the case of the scalar field has been given by Martin and Verdaguier (see Eq. 3.42 of Martin and Verdaguier, 1999).

5.2. The Metric Fluctuation Correlation Function

We can now use our expression for the stress tensor correlation function to find the correlation function for the passive metric fluctuations induced by vacuum fluctuations of the EM field. Let $h_{\mu\nu}$ be a classical metric perturbation due to the stress tensor $T_{\mu\nu}$. Define $\hbar_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ and impose the harmonic gauge condition, $(\partial_\nu \hbar^{\mu\nu} = 0)$. Then

$$\square \hbar_{\mu\nu} = -16\pi T_{\mu\nu} \quad (25)$$

in units in which $G = 1$, where G is Newton's constant. Let $G_r(x - x')$ be the retarded Green function which satisfies

$$\square G_r(x - x') = \delta(x - x'). \quad (26)$$

If there is no incoming gravitational radiation, $\hbar_{\mu\nu}(x)$ is given by

$$\hbar_{\mu\nu} = -16\pi \int d^4x_1 G_r(x - x_1) T_{\mu\nu}(x_1). \quad (27)$$

Now let $T_{\mu\nu}$ be the normal-ordered stress operator for the quantized EM field. Because here $T_\mu^\mu = 0$, we have $\hbar_{\mu\nu} = h_{\mu\nu}$. The metric fluctuation correlation function is now

$$\langle h^{\mu\nu}(x) h^{\rho\sigma}(x') \rangle = (16\pi)^2 \int d^4x_1 d^4x_2 G_r(x - x_1) G_r(x' - x_2) C_{(V)}^{\mu\nu\rho\sigma}(x_1, x_2). \quad (28)$$

We use Eqs. (22) and (24) in the above expression. The result may be written in terms of derivatives of the quantity

$$S = \ln^2[\mu^2(x - x')^2], \quad (29)$$

where μ is an arbitrary constant, using results such as

$$\square^2 S = -\frac{32}{[(x - x')^2]^2}. \quad (30)$$

Finally we perform a set of integrations by parts and assume that the surface terms can be ignored. (This assumption needs to be examined more carefully, and is a current topic of investigation.) More details of the calculation will be given in a later paper. The final result for the metric correlation function is

$$\begin{aligned} \langle h_{\mu\nu}(x)h_{\sigma\lambda}(x') \rangle = & -\frac{1}{60\pi^2} [4\partial_\mu\partial_\nu\partial_\sigma\partial_\lambda S + 2(g_{\mu\nu}\partial_\sigma\partial_\lambda + g_{\sigma\lambda}\partial_\mu\partial_\nu)\square S \\ & - 3(g_{\mu\sigma}\partial_\nu\partial_\lambda + g_{\mu\lambda}\partial_\nu\partial_\sigma + g_{\nu\sigma}\partial_\mu\partial_\lambda + g_{\nu\lambda}\partial_\mu\partial_\sigma)\square S \\ & + 3(g_{\mu\sigma}g_{\nu\lambda} + g_{\nu\sigma}g_{\mu\lambda})\square^2 S - 2g_{\mu\nu}g_{\sigma\lambda}\square^2 S]. \end{aligned} \quad (31)$$

This is a remarkably simple result. It is of special interest to note that the metric fluctuation correlation function is expressible as the total derivative of a scalar.

6. OPERATIONAL MEANING OF METRIC FLUCTUATIONS

Fluctuations of the space–time metric ultimately must be recorded by test particles or waves propagating in the fluctuating geometry. Let us first consider the use of a classical point test particle. In classical relativity, such a test particle moves on a geodesic in a fixed classical metric and can serve as giving operational meaning to the space–time geometry. If we now allow the metric to fluctuate, the geodesic equation becomes a Langevin equation and the test particles undergo Brownian motion (Kuo and Ford, 1993). We can express this Langevin equation as

$$\frac{du^\mu}{d\tau} = -\Gamma_{\alpha\beta}^\mu u^\alpha u^\beta - \gamma_{\alpha\beta}^\mu u^\alpha u^\beta, \quad (32)$$

where u^μ is the particle's four-velocity, $\Gamma_{\alpha\beta}^\mu$ is the connection due to the mean metric, and $\gamma_{\alpha\beta}^\mu$ is the linear correction to the connection due to the fluctuations. Thus

$$\langle \gamma_{\alpha\beta}^\mu \rangle = 0. \quad (33)$$

We may integrate this equation, and then calculate mean squared variations in the four-velocity in terms of the metric fluctuation correlation function, $\langle h^{\mu\nu}(x)h^{\rho\sigma}(x') \rangle$. Note that this correlation function is given in passive quantum gravity

by the generalization of Eq. (28):

$$\langle \hat{h}^{\mu\nu}(x) \hat{h}^{\rho\sigma}(x') \rangle = (16\pi)^2 \int d^4x_1 d^4x_2 G_r(x-x_1) G_r(x-x_2) C^{\mu\nu\rho\sigma}(x_1, x_2), \quad (34)$$

where now the full stress tensor correlation function $C^{\mu\nu\rho\sigma}(x_1, x_2)$ appears. This procedure allows us to calculate such quantities as the mean angular deflection or the mean time delay or advance due to the fluctuating metric.

Instead of a point particle, one might use classical waves as the probes of the fluctuating geometry (Hu and Shiokawa, 1998). In this case, one could write down a correction to a solution of a wave equation due to linearized metric perturbations, which plays a role analogous to the $\gamma_{\alpha\beta}^{\mu}$ term in Eq. (32). This term will produce fluctuations in the wave intensity at a given observation point. There is a need for more detailed model calculations to better understand both the test particle and the wave approaches to probing a fluctuating geometry.

7. VALIDITY OF THE SEMICLASSICAL THEORY OF GRAVITY

The semiclassical theory of gravity assumes a fixed space–time metric satisfying the semiclassical Einstein equation

$$G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle. \quad (35)$$

This equation is clearly an approximation which must fail at some point. First, it does not include any effects of the quantization of gravity itself, the active metric fluctuations. However, even if we restrict ourselves to situations where only the quantum effects of matter fields are included, Eq. (35) must fail when the passive metric fluctuations become too large. The question is, how large is too large?

Kuo and Ford (1993) suggested that a possible criterion could be based upon the quantity Δ defined in Eq. (8). If $\Delta \ll 1$, then the fractional fluctuations in the local energy density, as measured by $C^{\mu\nu\rho\sigma}(x, x')$ are small, and one expects the resulting metric fluctuations also to be small. However, if Δ is not small, then there are large local energy density fluctuations. Kuo and Ford took

$$\Delta \ll 1 \quad (36)$$

as a necessary condition for the validity of the semiclassical theory. This criterion has been criticized by Phillips and Hu (2000a,b) as being too strong. The latter authors calculate a quantity analogous to Δ , but involving smeared fields in the Minkowski vacuum state. They find that this quantity is of order one. Because one expects the semiclassical theory to be valid in Minkowski space–time, Phillips and Hu conclude that Eq. (36) is not a reliable criterion.

We wish to give an assessment both of the Kuo–Ford criterion and of Phillips and Hu criticism of it. First, it now seems that the Kuo–Ford criterion is at best

incomplete because it does not address the effects of the cross term. The radiation pressure fluctuations studied in Section 4.2 show that this term has physical reality and must contribute to quantum metric fluctuations. The extent of its contribution is not yet clear. However, in some model radiation pressure calculations for thermal states (Wu and Ford, 1999) and in the Casimir effect (Wu *et al.*, 2002), the cross term gives a larger contribution than does the fully normal ordered term. Furthermore, the real effect of both terms on metric fluctuations is measured by integrals along the worldlines of test particles rather than by local quantities.

However, the analysis of Phillips and Hu is open to the criticism that the quantities which they define are not directly observable. The type of averaging which is involved in a measurement of a fluctuating space–time by test particles is more of the form of that in Eq. (34) than of smearing field operators themselves. This leads us to the question of whether the pure vacuum term can have observable effects in Minkowski space–time. The metric fluctuation correlation function given in Eq. (31) is the total derivative of a scalar. This suggests that when one uses it to calculate the Brownian motion of test particles or the fluctuation in amplitude of a wave, the result can be cast into the form of a surface term by an integration by parts. However, surface terms can be made to vanish when quantities such as the wave amplitude are switched on in the past and off in the future. This is by no means a rigorous argument, but rather a heuristic suggestion that the pure vacuum term may not produce observable effects. This suggestion needs to be tested by more detailed analysis. If it is correct, then Phillips and Hu criticism of the Kuo–Ford criterion is muted.

This would still not necessarily mean that the Kuo–Ford criterion is a good measure of the effects of metric fluctuations. As noted above, it ignores the effects of the cross term. More generally, it now seems that any criterion for the validity of the semiclassical theory must be a nonlocal one. That is, it should involve integrals upon the worldlines of test particles. It is possible that one can have situations where there are large fluctuations on short time or distance scales, but which average out when measurements on longer scales are made.

If the vacuum term is indeed unobservable, then one must study in detail the combined effects of the normal ordered and the cross term on the Brownian motion of test particles. This also remains to be done. In the end, the validity of the semiclassical theory will probably depend on the question which one wishes to answer. If one is interested only in quantities averaged over scales large compared to the intrinsic scales defined by the quantum state, then the semiclassical theory may well give an accurate answer. However, if one poses a question about behavior on shorter scales, the fluctuations are more likely to be important. A useful analogy is the fluctuating mirror discussed in Section 4.2. If one is only interested in the average motion of the mirror, then Newton’s second law with the mean force is adequate. However, if one needs to know the position of the mirror to high accuracy, as in a sensitive interferometer, then the force fluctuations cannot be ignored.

In summary, it seems likely that the validity of the semiclassical approximation will depend upon several factors. First, it depends upon what question one is asking. This determines the level at which one decides that the effects of fluctuations around a mean geometry are negligible. Second, it can depend upon the choice of quantum state. We have seen that the fluctuations of the normal ordered term are minimized in a coherent state, but can be large in other states. Similarly, radiation pressure fluctuations are minimized in a photon number eigenstate, but can be significant in other states. Finally, the magnitude of fluctuation effects depends upon time and length scales, which can in turn depend upon the quantum state. Measurements which average over larger scales have a greater tendency to average out the effects of fluctuations than do those made on very short scales.

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